CHAOS 28, 061101 (2018)

Efficient disintegration strategies with cost constraint in complex networks: The crucial role of nodes near average degree

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(Received 17 March 2018; accepted 23 May 2018; published online 8 June 2018)

The study of network disintegration, including controlling disease spread and destroying terrorist organizations, has wide application scenarios and attracts many researchers. In this paper, we concentrate on the network disintegration problem with heterogeneous disintegration cost, where the disintegration cost to eliminate each node might be non-identical. We first put forward a disintegration cost model and an optimization model for disintegration strategy. Then, we analyze the hub strategy, leaf strategy, and the average degree strategy to investigate the nodes tendency of the optimal disintegration strategy. Numerical experiments in three synthetic networks and real-world networks indicate that the disintegration effect of hub strategy drops gradually when the disintegration cost changes from homogeneity to heterogeneity. For the situation of strong heterogeneity of disintegration cost of each node, average degree strategy achieves the maximum disintegration effect gradually. Also, taking another perspective, average degree strategy might enlighten efficient solutions to protect critical infrastructure through strengthening the nodes which are chosen by the average degree strategy. *Published by AIP Publishing.* https://doi.org/10.1063/1.5029984

Various real-world systems can be studied by employing complex network methods, examples including road networks, airline networks, Communication networks, collaboration networks, etc. A majority of these networks are crucial to human life, and many studies have concentrated on methods to enhance their reliability. However, there are still many networks that are harmful to human life, such as disease spread networks and terrorist organizations. The essential purpose of this work is to seek efficient disintegration strategies in network disintegration problem.

I. INTRODUCTION

In modern society, the targets and the connection forms of the targets always form a complex network in network disintegration problem, such as power grids,[1] communication networks,[2] and transportation networks.[3] Thus, it is necessary to be able to determine the disintegration effect that may be inflicted upon the networks by network disintegration strategy. Apparently, network disintegration strategy has broad applications; moreover, examples would include destroying terrorist networks,[4] remedying financial crises,[5] impeding the spread of rumors,[6] and blocking cancer networks,[7] which have received growing attention.

Regarding network disintegration, the challenge is to identify the optimal set of nodes such that their removal could achieve the maximum disintegration effect. An efficient way of disintegrating a network is according to the structural properties of nodes, the so-called centrality-based disintegration strategies. The most straightforward and fundamental disintegration strategy is the degree centrality strategy,[8] in which nodes are disintegrated by the declining order of the original degree of network, nevertheless, it is less relevant since the removal of a node having a few high degree neighbors may have much more destructive disintegration effect than a node having a larger number of low degree neighbor.[9,10] Although some well-known global properties such as betweenness centrality,[11,12] coreness,[13] and subgraph centrality,[14] which can achieve better disintegration effect, due to the very high computation complexity, they are not easy to implement in large networks. To achieve better disintegration effect, other methods were proposed. For example, Chen *et al.*[15] put forward the equal graph partitioning (EGP) immune approach. The fundamental intention of EGP focused on disintegrating the network into some connected components with equal dimension. Deng *et al.*[16] presented an optimization frame to seek an optimal disintegration strategy and applied the intelligent optimization algorithm to solve the optimization model. Besides, it has received more attention on imperfect information strategy. For instance, Li *et al.*[17] concentrated on the network disintegration approaches with incomplete network structure. The study assumed that few parts of the network structure could be acquired. Tan *et al.*[18] put forward link prediction method to analyze disintegration approach with incomplete information.

Nevertheless, regarding the present studies, a majority of studies on network disintegration problem supposed that the disintegration cost to remove every node is equal. Actually, the disintegration cost of each node might be heterogeneous in real-world scenarios. On the one hand, some researchers have studied the counterterrorism problem considering the budget constraint problem.[19] The situation is similar in suppressing the epidemic spreading within a fixed vaccination budget.[20] On the other hand, it is realistic to assume that

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different target has a different value for both the attacker and the defender. For example, New York City would be valued much higher than a desert area.[21] Many researchers assumed that the defender and the attacker have the same target valuations;[22,23] therefore, it is easy to know that the disintegration cost assigned to critical infrastructure would be much more than some regular auxiliary facilities. In this study, we will concentrate on the network disintegration problem considering the heterogeneous costs. Then, we present three typical disintegration strategies: Hub Strategy (HS), Leaf Strategy (LS), and Average Degree Strategy (AS), to compare the corresponding disintegration effect. In the traditional view, nodes with the high attribute value would be selected to be removed preferentially. However, hub nodes may not be the best choice concerning the heterogeneous disintegration costs.

This study is organized as follows: The disintegration cost model and the optimization model for network disintegration problem are shown in Sec. II. In Sec. III, we introduce the measure of network performance. The experiments in three types of model networks and real-world networks are presented in Secs. IV and V, respectively. Finally, we show the conclusions and discussions in Sec. VI.

II. COST MODEL AND OPTIMIZATION MODEL FOR DISINTEGRATION STRATEGY IN COMPLEX NETWORKS

Complex networks can be formed as an undirected graph $G(V, E)$, in which V is the node set, and $E \subseteq V \times V$ is the edge set. Let $N = |V|$ be the number of nodes and $W = |E|$ be the number of edges, respectively. Denote by $A(G) = (a_{ij})_{N\times N}$ the adjacency matrix of G, in which $a_{ij} = a_{ji} = 1$ once v_i and v_j are connected. Besides, let k_i be the degree of node v_i. The value of k_i is the number of adjacent edges of node v_i.

In this paper, the target of removal approach focuses on node removal, and we assume that the adjacent edges will be removed when the corresponding node is removed. Denote by c_i the disintegration cost of corresponding node v_i. Note that many researchers assumed that the defender and the attacker have the same target valuations;[22,23] therefore, it is easy to know that the disintegration resource assigned to critical infrastructure would be much more than some regular auxiliary facilities. Thus, the linear function might not be sensitive enough to define the disintegration cost of every target. In this paper, we suppose that the disintegration cost c_i of each node is a function of node property r_i

$$c_i = r_i^p, \tag{1}$$

in which we call $p \geq 0$ the cost-sensitive parameter. We can choose one of the node properties as r_i, examples including the degree and the betweenness. Specifically, the higher value of p indicates that the disintegration cost is more sensitive.

In real-world scenarios, the total disintegration budget is limited. Denote by $\hat{C}$ the cost constraint. In this paper, we define it in the following form:

$$\hat{C} = \alpha \sum_{i=1}^{N} c_i = \alpha \sum_{i=1}^{N} r_i^p, \tag{2}$$

in which we call $\alpha \in [0, 1]$ the cost-constraint parameter. The higher value of α indicates that the constraint of disintegration budget is looser.

Let $\hat{V} \subseteq V$ be the removed nodes set. Denote by $\hat{G} = (V - \hat{V}, \hat{E})$ the network after disintegration. Denote by $n = |\hat{V}|$ the number of removed nodes. Let $X = [x_1, x_2, \ldots, x_N]$ be the disintegration strategy, in which $x_i = 1$ if $v_i \in \hat{V}$, otherwise $x_i = 0$. It is easy to know

$$n = \sum_{i=1}^{N} x_i. \tag{3}$$

Denote by $C_X = \sum_{v_i \in \hat{V}} c_i$ the cost of disintegration strategy X. Then, we obtain

$$C_X = \sum_{v_i \in \hat{V}} c_i = \sum_{i=1}^{N} x_i c_i = \sum_{i=1}^{N} x_i r_i^p. \tag{4}$$

Let Γ be the measure function of network performance. In this paper, we assume that if $G_1 = (V_1, E_1)$ is a subgraph of $G_2 = (V_2, E_2)$, i.e., $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$, then $\Gamma(G_1) \leq \Gamma(G_2)$. The monotonicity assumption guarantees that the measure function declines after node removal. In this paper, we define the disintegration effect as the deterioration of network performance $\Phi(X) = \Gamma(G) - \Gamma(\hat{G}) \geq 0$. The goal of the optimization model is to identify the optimal solution X^*, which could maximize the disintegration effect. Therefore, the objective function is shown as follows:

$$\max \Phi(X = [x_1, x_2, \ldots x_N]) \\ s.t. \begin{cases} C_X \leq \hat{C} \\ x_i = 0 \text{ or } 1, i = 1, 2, \ldots, N. \end{cases} \tag{5}$$

III. THE MEASURE FUNCTION OF NETWORK PERFORMANCE

In this paper, we consider the natural connectivity to evaluate the disintegration effect because the value of natural connectivity declines strictly monotonically once the nodes are removed. The natural connectivity can be described in the following form.

Denote by $v_0 e_1 v_1 e_2 \ldots e_l v_l$ a walk of length l, in which $v_i \in V$ and $e_i = (v_{i-1}, v_i) \in E$. We call a walk is closed once $v_0 = v_l$. Closed walks precisely related to the subgraphs of a graph. For example, it stands for an edge if the length of the closed walk is 2. The number of closed walks is a critical indicator in network science. For example, Estrada and Rodriguez-Velazquez have studied vertex centrality[14] and network bipartivity[24] concerning the number of closed walks. In the previous study, we have revealed the positive correlation between the number of closed walks and the redundancy of alternative paths. Thus, it is clear that the number of closed walks can be regarded as a measure function of disintegration effect.[25–27] Concerning that shorter

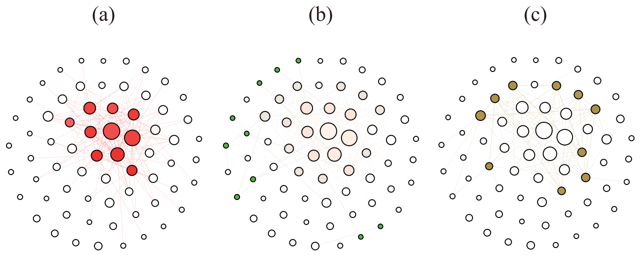

FIG. 1. The visualization of three typical disintegration strategies. The size of each node is proportional to its degree; the solid red circles indicate the removed nodes in Hub Strategy (a), the solid green circles indicate the removed nodes in Leaf Strategy (b), and the solid yellow circles indicate the removed nodes in Average Degree Strategy (c).

closed walks play a more important role on the redundancy than longer closed walks, we define a weighted sum of numbers of closed walks $S = \sum_{l=0}^{\infty} n_l/l!$, in which we call n_l the number of closed walks of length $l > 0$, and it is easy to know that $n_0 = N$. This scaling guarantees that the weighted amount does not deviate; besides, it also means that the weighted amount of numbers of closed walks S could be acquired from the power of the adjacency matrix

$$n_l = \sum_{i=1}^{N} \lambda_i^l = trace(A^l) = \sum_{i=1}^{N} \lambda_i^l, \qquad (6)$$

in which λ_i is the ith largest eigenvalue of the adjacency matrix $A(G)$. Then, we obtain

$$S = \sum_{l=0}^{\infty} \frac{n_l}{l!} = \sum_{l=0}^{\infty} \sum_{i=1}^{N} \frac{\lambda_i^l}{l!} = \sum_{i=1}^{N} \sum_{l=0}^{\infty} \frac{\lambda_i^l}{l!} = \sum_{i=1}^{N} e^{\lambda_i}. \qquad (7)$$

Note that the weighted amount of numbers of closed walks S would be a large number for the large value of N; the natural connectivity is then described as an average eigenvalue of the graph in the following form:

$$\bar{\lambda} = \ln\left(\frac{S}{N}\right) = \ln\left(\frac{1}{N}\sum_{i=1}^{N} e^{\lambda_i}\right). \qquad (8)$$

In Ref. 15. Chen *et al.*, it has shown that the natural connectivity declines strictly monotonically once the nodes are removed. Therefore, lower values correspond to more destructive attack strategies.

IV. THREE TYPICAL DISINTEGRATION STRATEGIES IN MODEL NETWORK

A. Three typical disintegration strategies

The optimization model is zero-one integer programming problem. From the previous study, we can apply precise mathematical programming approaches to solve it, including the branch and bound algorithm.[28] Nevertheless, an objective function with an explicit formulation is needed when we utilize precise mathematical programming approaches. In this work, we utilize $\Phi(X) = \Gamma(G) - \Gamma(\hat{G})$

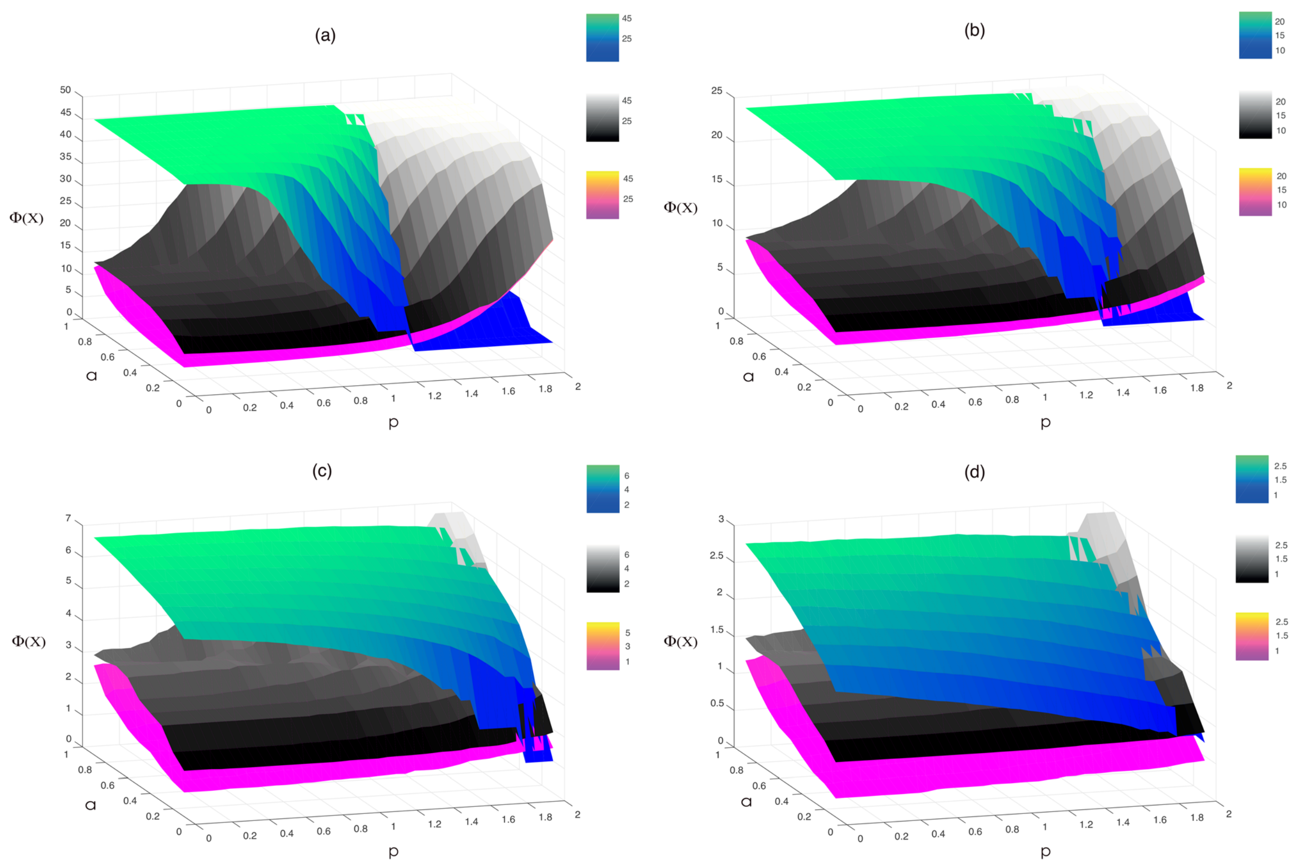

FIG. 2. The disintegration effect of three typical strategies, i.e., Hub Strategy (green-blue color bar), Leaf Strategy (yellow-pink color bar), and Average Degree Strategy (dark gray color bar) in random scale-free networks with degree distribution $p(k) = (\gamma - 1)m^{\gamma-1}k^{-\gamma}$, where (a) $N = 1000$, $\gamma = 2.0$; $m = 3$, (b) $N = 1000$, $\gamma = 2.5$; $m = 3$, (c) $N = 1000$, $\gamma = 3.0$; $m = 3$, and (d) $N = 1000$, $\gamma = 3.5$, and $m = 3$.

as the objective function of the optimization model, in which Γ is the value of natural connectivity. Although we can compute Γ when the disintegration strategy X is given, there is no explicit formulation for Γ as a function of the disintegration strategy X. Therefore, the precise mathematical programming approaches are not appropriate for the problem. Thus, we contemplate by utilizing three typical disintegration strategies for the optimization problem to investigate the nodes selectivity of the optimal disintegration strategies.

Before analyzing the disintegration effect among these strategies, we first present the visualization of three typical disintegration strategies in Fig. 1. In the following part, we select the degree to be the node property r_i.

(i) **Hub Strategy (HS):** Nodes are eliminated by the declining order of the degree k_i, which indicates that the hub nodes will be removed preferentially.

(ii) **Leaf Strategy (LS):** Nodes are eliminated by the upward order of the degree k_i, which indicates that the "leaf" nodes will be removed preferentially.

(iii) **Average Degree Strategy (AS):** Nodes near average degree will be eliminated preferentially. Denote the average degree by $\langle k \rangle$. Let $d = (k_1, k_2, k_3, \ldots k_N)$ be the degree sequence. Denote by $\theta_i = |k_i - \langle k \rangle|$ the deviation value of the degree of node v_i from the average degree. Therefore, nodes near average degree will be removed preferentially means that nodes are removed by the upward sequence of θ_i.

B. Experiments in synthetic networks

In this part, we will display three typical disintegration strategies in scale-free networks, Erdös-Rényi (ER) random networks, and Newman-Watts (NW) small-world networks. (i) Scale-free network: The most remarkable feature in a scale-free network is the relative commonness of nodes with a degree that significantly exceeds the average. We generate random scale-free networks with degree distribution $p(k) = (\gamma - 1)m^{\gamma-1}k^{-\gamma}$ using the configuration model,[29,30] where γ is the power-law exponent and m is the smallest degree. (ii) ER random network: Not all real-world networks are scale-free. Therefore, we also perform the method for the Erdös-Rényi random network (ER network).[31] Erdös and Rényi presented a simple model of the random network. Take some number N of nodes and connect each pair with probability $p_{connection}$. Erdös and Rényi called this network model as the $G_{N,p}$. (iii) NW small-world network: First, start with a nearest-neighbor coupled network with N nodes placed in a ring, in which every node v_i is connected to its neighbors, $i = 1, 2, \ldots, k/2$, with K being even. Then, add with probability $p_{connection\_NW}$ an edge between a pair of nodes.[32] Moreover, we use the natural connectivity as the measure function of network performance Γ.

To display the influence of the heterogeneity cost and the cost constraint among three typical disintegration strategies in the scale-free network, we show in Fig. 2 the effect of disintegration strategy $\Phi(X)$ as functions of the cost-sensitive

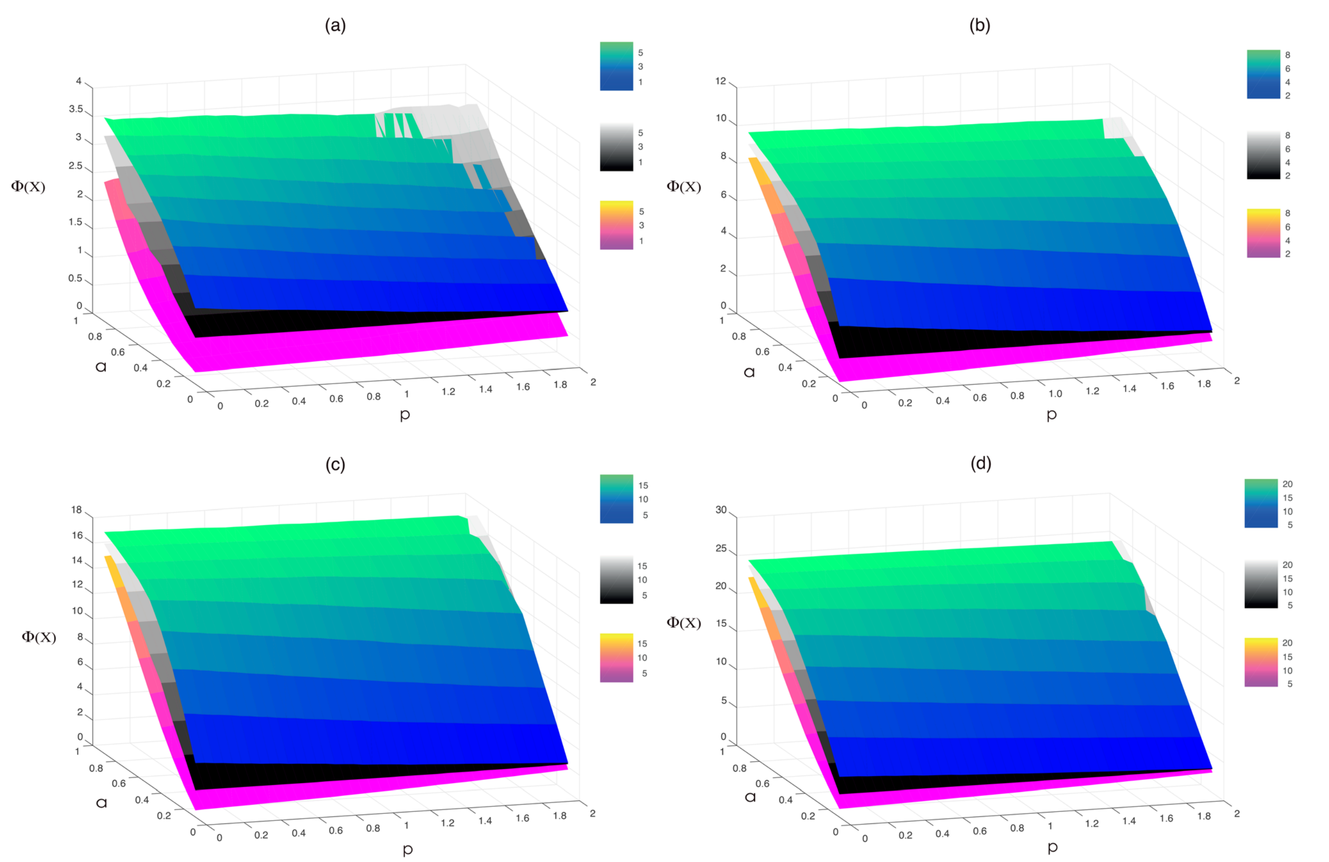

FIG. 3. The disintegration effect of three typical strategies, i.e., Hub Strategy (green-blue color bar), Leaf Strategy (yellow-pink color bar), and Average Degree Strategy (dark gray color bar) in Erdös-Rényi networks. (a) $N = 1000$, $p_{connection} = 0.008$; (b) $N = 1000$, $p_{connection} = 0.016$; (c) $N = 1000$, $p_{connection} = 0.024$; (d) $N = 1000$, $p_{connection} = 0.032$.

parameter p and the cost-constraint parameter α with different values of γ. In Fig. 2, the higher value of the $\Phi(X)$, the more destructive the disintegration strategy is. From Figs. 2(a)–2(d), the most interesting observation is that AS can maximize the disintegration effect within an area. Significantly, HS can achieve the disintegration effect better than the other two typical disintegration strategies in the most cases. In the rest cases, the dominance of HS fades progressively, while the disintegration cost changes from homogeneity to heterogeneity. As shown in Fig. 3, the situation is similar in ER network. HS is still the best choice when the disintegration costs are not sensitive, and the disintegration effect of AS surpasses HS when disintegration cost becomes heterogeneous. Moreover, we show in Fig. 4 the disintegration effect in NW small-world networks. From Figs. 4(a) and 4(b), it is clear that AS can achieve optimal disintegration effect within large areas when the cost constraint is loose. As we know, the NW model will be the original nearest-neighbor coupled network with $p_{connection\_NW}=0$, and it becomes a globally coupled network if $p_{connection\_NW}=1$.[33] Although the disintegration effect of AS decreases when the network changes from a nearest-neighbor coupled network to random network, it still remains the maximum disintegration effect in a small area in Fig. 4(d).

Moreover, we show the vertical projection of the disintegration effect in Figs. 5–7. We observe that there is a watershed between the disintegration effect of HS and AS, especially in Figs. 5 and 7. As shown in Figs. 5, 6, and 7(d), if the disintegration cost c_i is not particularly sensitive, HS is the most destructive strategy. For the situation of strong heterogeneity of disintegration cost ($p>1.4$), AS has performed better than HS gradually. The situation is slightly different in NW small-world networks in Figs. 7(a)–7(c). AS takes over the optimal disintegration strategy when the cost constraint is loose enough, and no matter the disintegration cost is sensitive or not, e.g., $\alpha=0.8$, and $\alpha=0.9$.

In general, attackers want to allocate resources to hub targets to maximize the total expected disintegration effect, whereas the defenders want to hide the hub nodes as much as possible to protect a network. Therefore, hub nodes are always hard to be destroyed. On the other hand, although leaf nodes are often easy to be accessed, LS shows the worst performance from the previous results. Thus, the AS seems to be the optimal strategy among them. Let us imagine that average degree nodes play a role like a fence between hub nodes and leaf nodes; therefore, we call the disintegration effect of AS as the "fence effect" in this paper. To investigate the details about the "fence effect," we show in Fig. 8 the details about the disintegration effect $\Phi(X)$ of AS as a function of the cost-sensitive parameter p in three model networks. As shown in Fig. 8(a), it is clear that the "fence effect" is more destructive in the scale-free network when the disintegration cost is more heterogeneous and the cost constraint is looser. Especially, if the disintegration cost c_i is superlinearly related to the degree d_i ($p>1$), the "fence effect" increases dramatically no matter how much

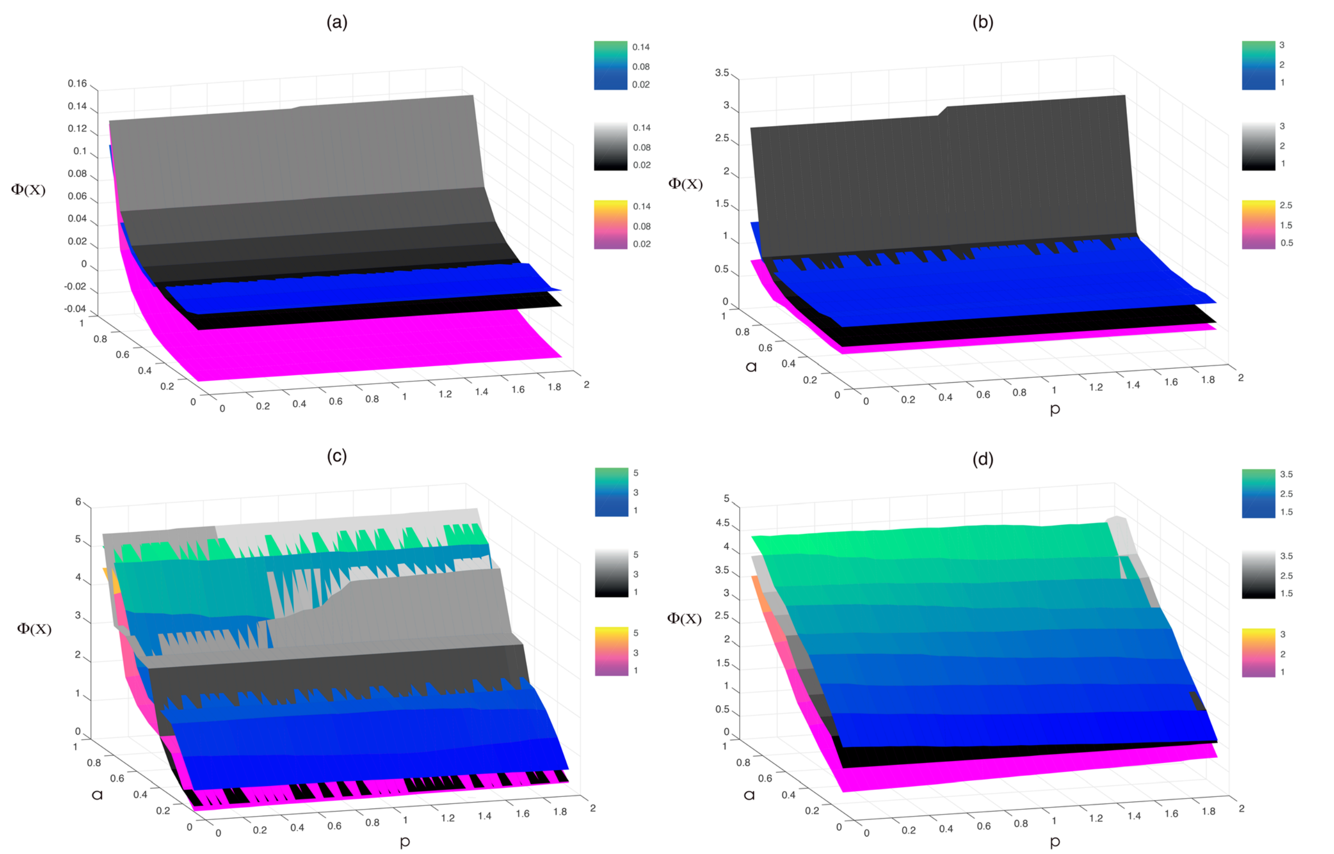

FIG. 4. The disintegration effect of three typical strategies, i.e., Hub Strategy (green-blue color bar), Leaf Strategy (yellow-pink color bar), and Average Degree Strategy (dark gray color bar) in NW small-world networks. (a) $N=1000$, $p_{connection\_NW}=0.001$; (b) $N=1000$, $p_{connection\_NW}=0.01$; (c) $N=1000$, $p_{connection\_NW}=0.1$; (d) $N=1000$, $p_{connection\_NW}=0.5$.

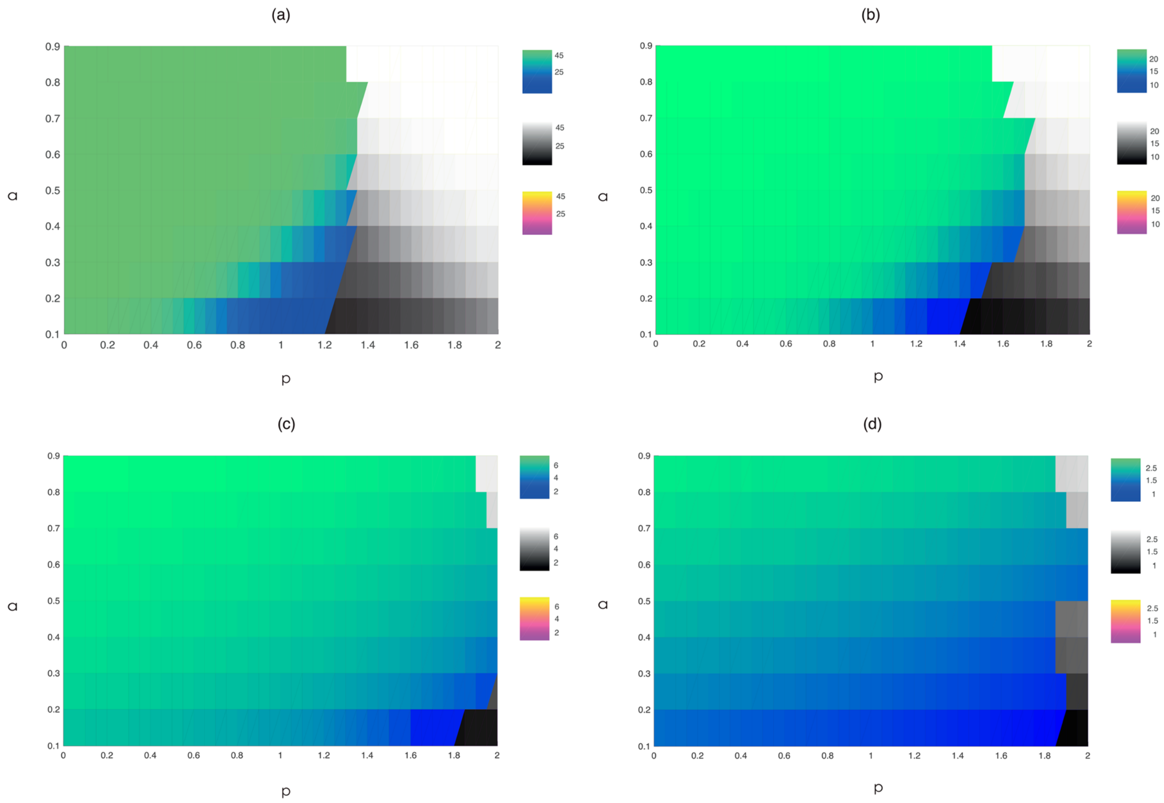

FIG. 5. The vertical projection of Fig. 2 is shown for auxiliary observation, i.e., Hub Strategy (green-blue color bar), Leaf Strategy (yellow-pink color bar), and Average Degree Strategy (dark gray color bar). (a) $N = 1000$, $\gamma = 2.0$, and $m = 3$; (b) $N = 1000$, $\gamma = 2.5$, and $m = 3$; (c) $N = 1000$, $\gamma = 3.0$, and $m = 3$; (d) $N = 1000$, $\gamma = 3.5$, and $m = 3$.

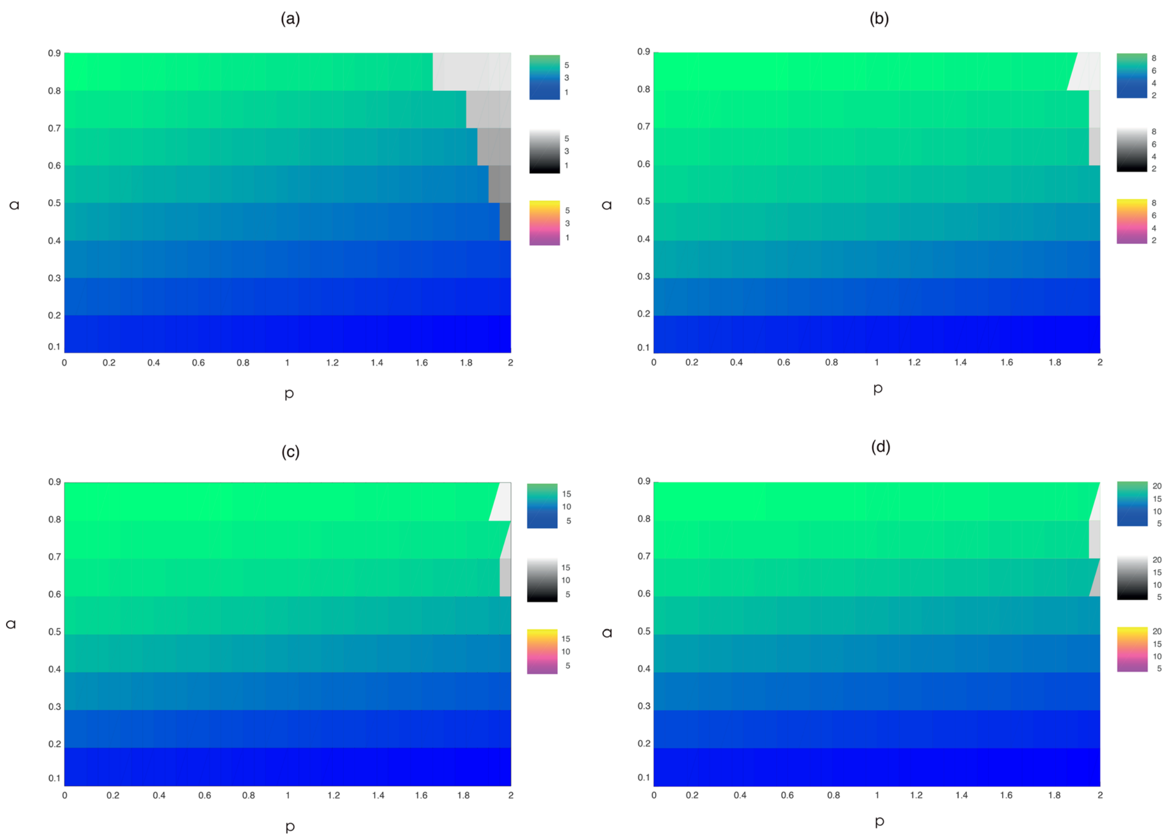

FIG. 6. The vertical projection of Fig. 3 is shown for auxiliary observation, i.e., Hub Strategy (green-blue color bar), Leaf Strategy (yellow-pink color bar), and Average Degree Strategy (dark gray color bar). (a) $N = 1000$, $p_{connection} = 0.008$; (b) $N = 1000$, $p_{connection} = 0.016$; (c) $N = 1000$, $p_{connection} = 0.024$; and (d) $N = 1000$, $p_{connection} = 0.032$.

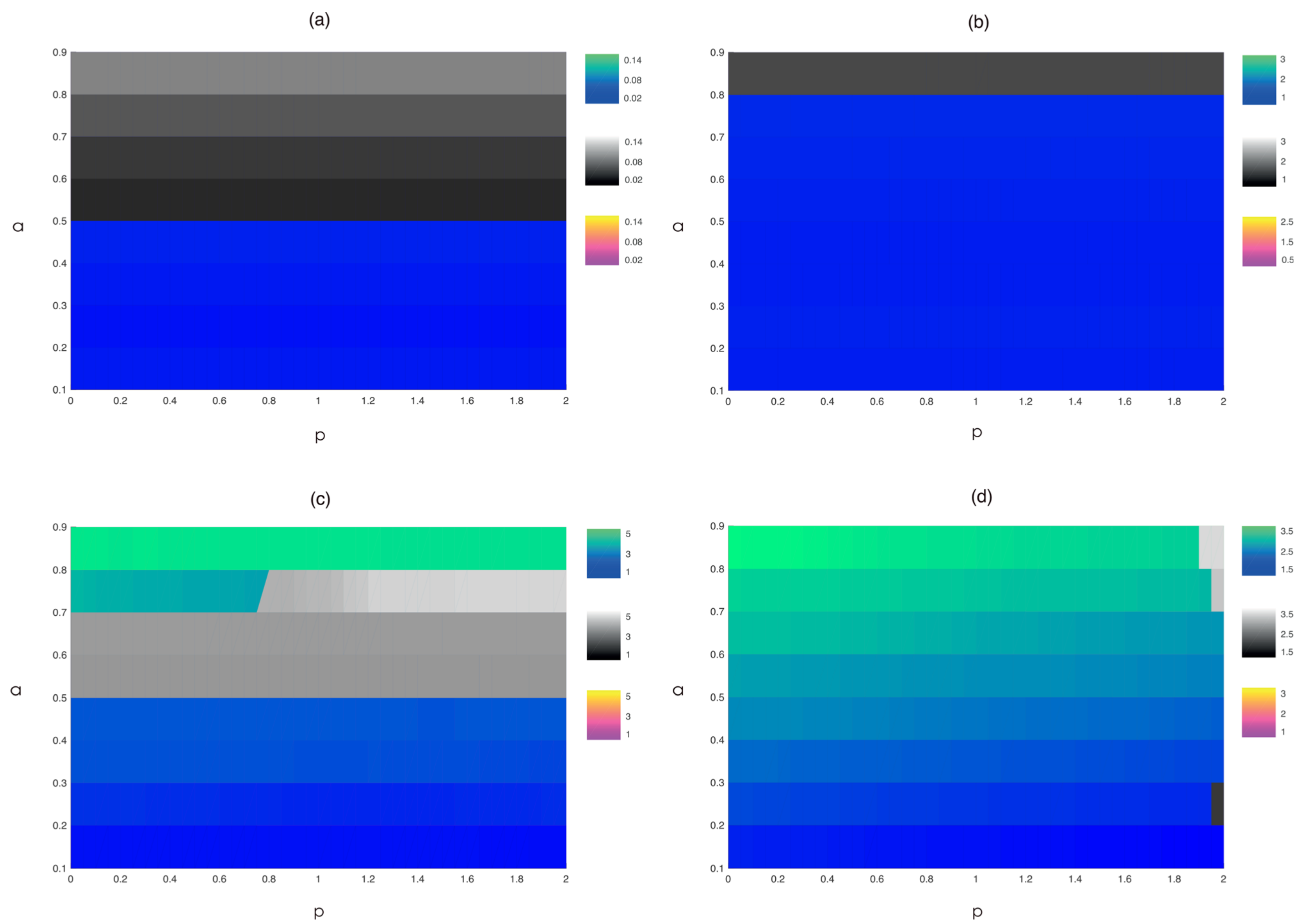

FIG. 7. The vertical projection of Fig. 4 is shown for auxiliary observation, i.e., Hub Strategy (green-blue color bar), Leaf Strategy (yellow-pink color bar), and Average Degree Strategy (dark gray color bar). (a) $N = 1000$, $p_{connection\_NW} = 0.001$; (b) $N = 1000$, $p_{connection\_NW} = 0.01$; (c) $N = 1000$, $p_{connection\_NW} = 0.1$; and (d) $N = 1000$, $p_{connection\_NW} = 0.5$.

disintegration cost constraint is, which indicates that average degree nodes show better disintegration effect in the case of heterogeneous cost ($p > 1$). From Fig. 8(b), the "fence effect" constantly remains when $p < 1$, and it increases slowly when $p > 1$ in the ER network. As shown in Fig. 8(c), the disintegration effects of AS in NW small-world networks are similar to ER random networks. The "fence effect" tends to increase slowly with increasing value of the cost-sensitive parameter p.

To investigate the "fence effect" in depth, we show the number of removed nodes n with various parameters settings of p and α in Fig. 9. We observe an uptrend of n towards the large values of p and α in Figs. 9(b), 9(d), and 9(f). It suggests that more nodes near average degree will be removed once the disintegration cost is more heterogeneous and the cost constraint is looser, which is in good agreement with the previous result that the "fence effect" shows a clear upward trend in three model networks. Conversely, the number of removed nodes of HS shows a downward trend dramatically in Figs. 9(a), 9(c), and 9(e), which is corresponding to the declining trend of disintegration effect of HS. From Figs. 9(c)–9(f), although the trends of the number of removed nodes are more gradual in ER networks and NW small-world networks, the uptrend of AS and downtrend of HS are also similar to the situations in scale-free networks, which is corresponding to the trend of disintegration effect in ER networks and NW small-world networks, respectively. The results are similar in the rest synthetic networks that we discussed above.

C. The accuracy of node attribution in average degree strategy

To investigate whether hubs are still removed when we employ AS, we quantify this problem by examining the deviation of the degree of removed nodes from the average degree. Denote by k_i the degree of node v_i. Let $\langle k \rangle$ be the network's original average degree. We denote the removed nodes set by $\hat{V} \subseteq V$ and the number of removed nodes by $n = |\hat{V}|$. Denote by $X = [x_1, x_2, \ldots x_N]$ the disintegration strategy, in which $x_i = 1$ if $v_i \in \hat{V}$, otherwise $x_i = 0$. Denote by Θ_X the measure function to quantify the deviation of the degree of removed nodes from the average degree. The form of Θ_X is as follows:

$$\Theta_X = \frac{\sum_{v_i \in \hat{V}} \theta_i}{n} = \frac{\sum_{i=1}^{N} x_i \theta_i}{n} = \frac{\sum_{i=1}^{N} x_i |k_i - \langle k \rangle|}{n}. \tag{9}$$

We calculated the values of Θ_X for twelve model networks and showed the results of Θ_X in Tables I–III, respectively. For example, note that AS is the optimal disintegration

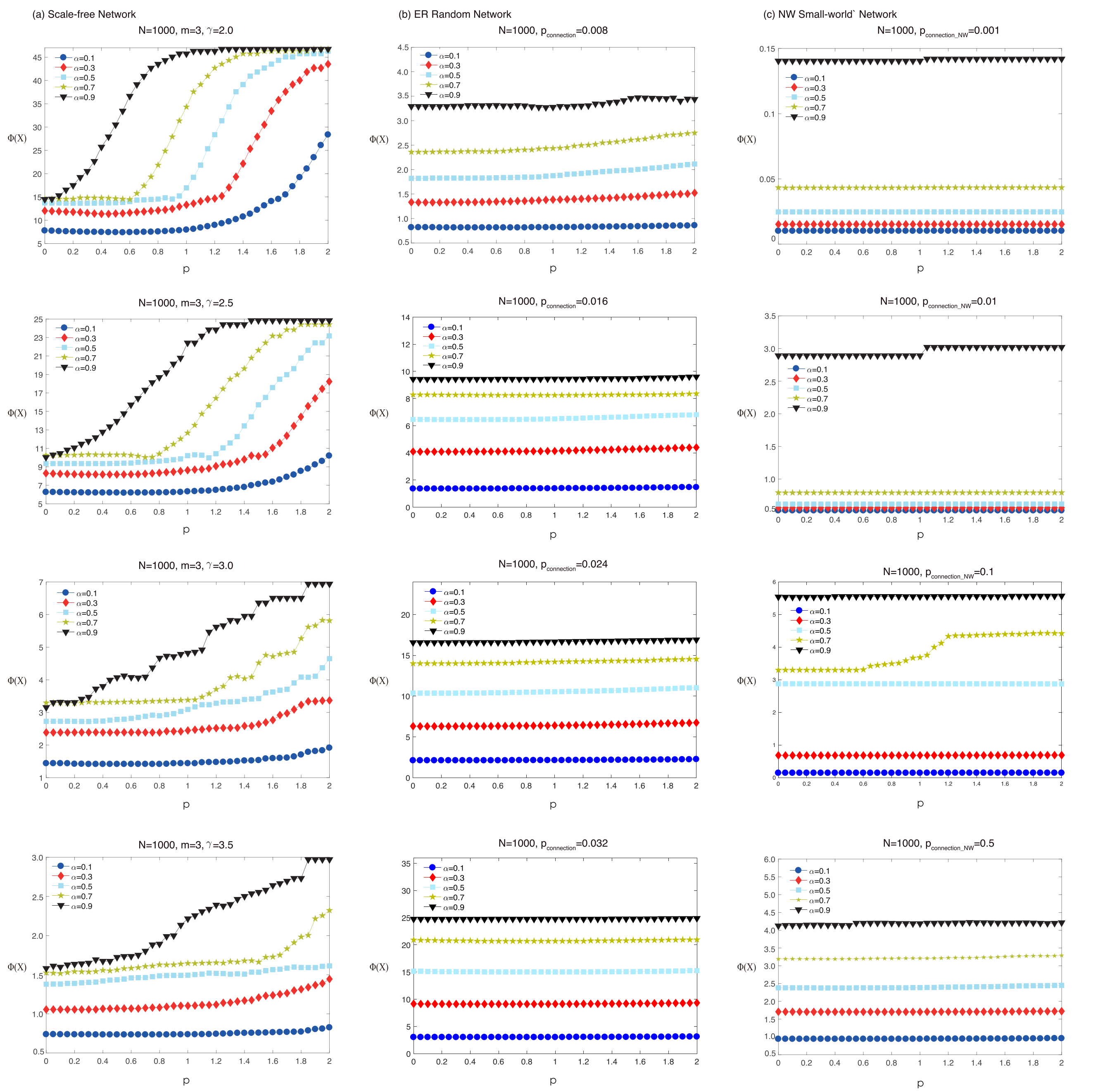

FIG. 8. The disintegration effect $\Phi(X)$ of AS as a function of the cost-sensitive parameter in (a) Scale-free network, (b) ER network, and (c) NW small-world network. The original network and data are the same as those we used in Figs. 2–4, respectively.

strategy when $p = 1.5$ and $\alpha = 0.1$ from Figs. 2(b) and 5(b), while the corresponding value of Θ_X is only 0.97 in Table I. As shown in Table I, we can see that most values of Θ_X are within the interval [0,3]. The values of Θ_X are relatively quite low because there are more than 100 nodes whose degrees are larger than 14 and the average degree is around 8. Especially, when the number of removed nodes is greater than almost 800, the value of Θ_X is just more than 3. The nice results owe to the technique that we generate the AS, where the nodes are deleted by the upward sequence of θ_i. For ER network and NW small-world network, the values of Θ_X are even lower, in which most of the results are below 1. The situations are similar to the rest of synthetic networks that we discussed in Sec. IV B. Therefore, it is quite clear that AS will not remove hubs unless the cost constraint is loose enough which leads to the fact that most nodes would be removed including hubs, and the higher disintegration effect of AS is also not related to the reason that many hubs are still removed when we employ AS.

V. EXPERIMENTS IN REAL-WORLD NETWORK

The study of network disintegration with heterogeneous cost is crucial for many real-world networks. To show the applicability of the presented framework, we implement experiments in three real-world networks: (i) the Food web of south Florida during the wet season,[34] (ii) the Political blogosphere (PB) network,[35] and (iii) the network of the US

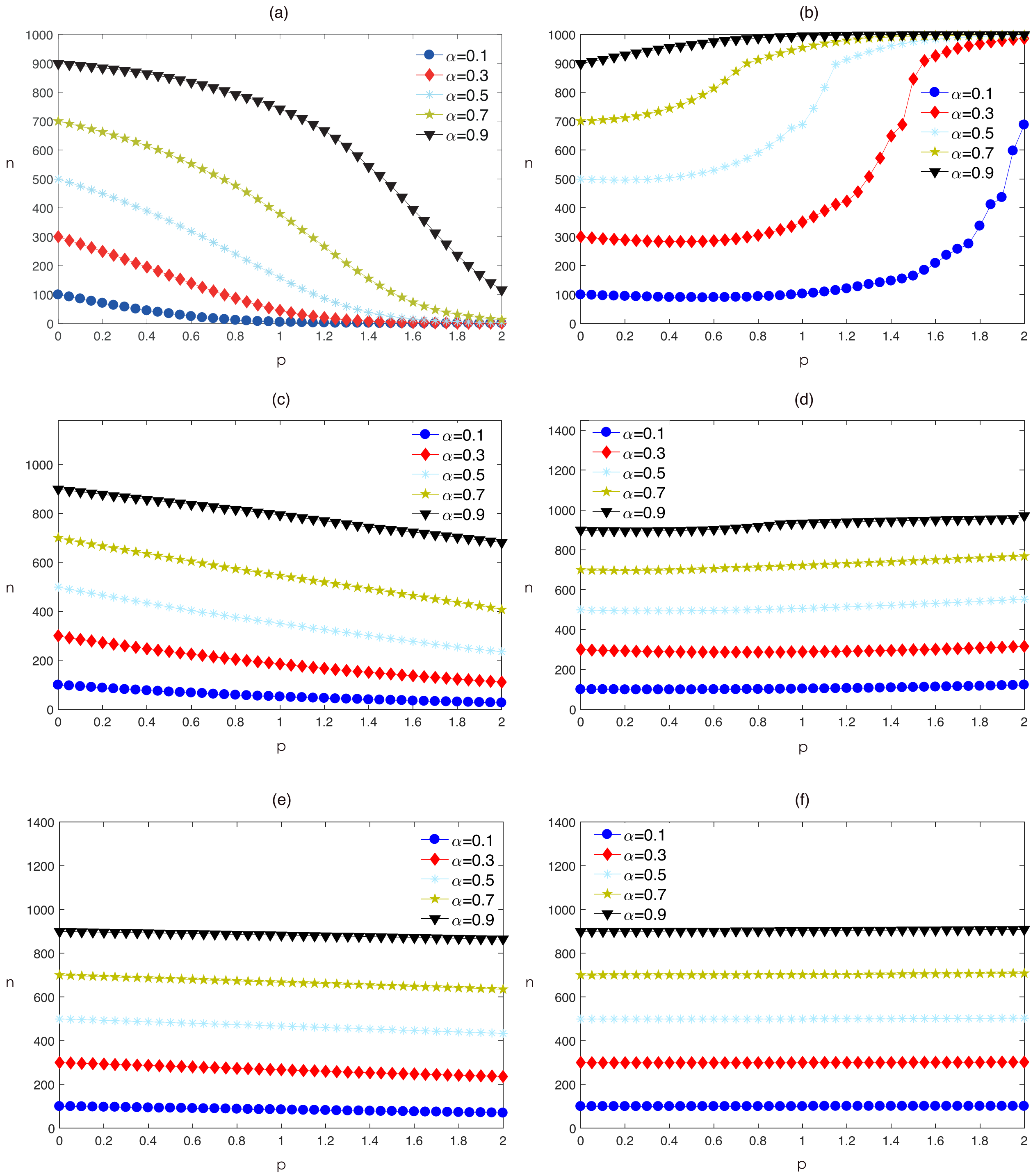

FIG. 9. The number of removed nodes of (a) HS in the scale-free network, (b) AS in the scale-free network, (c) HS in the ER network, (d) AS in the ER network, (e) HS in the NW small-world network, and () AS in the NW small-world network, The original network and data are the same as those we used in Figs. 2(b), 3(a), and 4(c), respectively.

TABLE I. The value of Θ_X in the scale-free network, and the original network is the same as the one we used in Fig. 2(b), where $N = 1000$, $\gamma = 2.5$, and $m = 3$.

α	p = 0	p = 0.25	p = 0.5	p = 0.75	p = 1	p = 1.25	p = 1.5	1.75	2
0.1	0.64	0.60	0.58	0.59	0.65	0.76	0.97	1.53	2.94
0.3	1.67	1.60	1.58	1.67	1.89	2.28	3.36	3.90	4.47
0.5	2.45	2.44	2.50	2.67	2.94	3.58	4.15	4.70	5.06
0.7	2.98	3.03	3.19	3.48	3.81	4.37	4.89	5.16	5.29
0.9	3.47	3.64	3.96	4.44	4.97	5.29	5.46	5.46	5.46

TABLE II. The value of Θ_X in the ER network, and the original network is the same as the one used in Fig. 3(a), where $N = 1000$ and $p_{connection} = 0.008$.

α	p = 0	p = 0.25	p = 0.5	p = 0.75	p = 1	p = 1.25	p = 1.5	1.75	2
0.1	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
0.3	0.43	0.42	0.41	0.41	0.41	0.42	0.43	0.44	0.45
0.5	0.72	0.71	0.71	0.72	0.73	0.75	0.78	0.80	0.83
0.7	1.09	1.09	1.09	1.11	1.14	1.16	1.19	1.21	1.24
0.9	1.56	1.55	1.56	1.60	1.66	1.69	1.71	1.73	1.79

TABLE III. The value of Θ_X in the NW small-world network, and the original network is the same as the one we used in Fig. 4(c), where $N = 1000$ and $p_{connection\_NW} = 0.1$.

α	p = 0	p = 0.25	p = 0.5	p = 0.75	p = 1	p = 1.25	p = 1.5	1.75	2
0.1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.5	0.07	0.07	0.07	0.07	0.07	0.07	0.08	0.08	0.08
0.7	0.34	0.34	0.34	0.34	0.34	0.34	0.34	0.34	0.35
0.9	0.49	0.49	0.50	0.50	0.50	0.50	0.50	0.51	0.51

TABLE IV. Basic statistics of real networks, N and W, are the numbers of nodes and links. $\langle k \rangle$ is the average degree; C is the clustering coefficient; r is the assortativity; and $\langle l \rangle$ is the average shortest distance.

Networks	N	W	$\langle k \rangle$	C	r	$\langle l \rangle$
Food web	128	2106	32.906	0.312	−0.111	1.73
PB	1222	16714	27.36	0.36	−0.221	3.65
Usairports	1574	28 236	35.8	0.384	−0.113	3.14

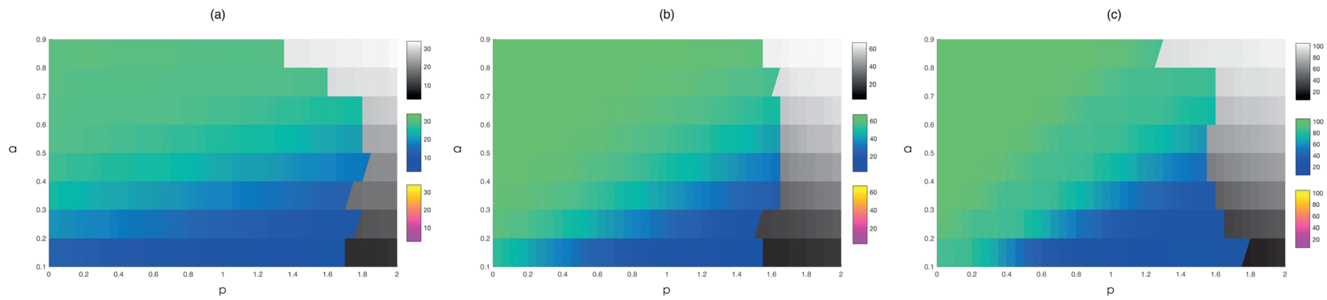

FIG. 10. The disintegration effect of three typical strategies in three real-world networks, i.e., (a) Food web, (b) Political blogosphere network and (c) the network of the US air transportation system. The lighter color indicates the higher value of $\Phi(X)$: Hub Strategy (green-blue color bar), Leaf Strategy (yellow-pink color bar), and Average Strategy (dark gray color bar).

air transportation system (http://toreopsahl.com/datasets/usairports). Basic statistics of three real-world networks are displayed in Table IV. As shown in Table IV, all of the networks are well connected, with high clustering coefficients and short average distances.

We simulate the disintegration effect of three disintegration strategies on these networks, and results are shown in Fig. 10. All three networks exhibit a similar pattern with the synthetic networks: HS can achieve the disintegration effect better than the other two typical disintegration strategies in the most cases. In the rest cases, the dominance of HS fades progressively; and AS becomes the best choice when the disintegration cost is from homogeneous into heterogeneous, and LS remains the worst strategy. It is critical to observe that all the three real-world networks have a "fence effect" area where AS achieves maximum disintegration effect entirely among three strategies. In general, critical infrastructure implies more disintegration cost, which suggests that the disintegration cost is heterogeneous. Thus, the AS could work better within the problem of network disintegration than HS and LS according to the results above. Taking another perspective, AS may also enlighten efficient methods to protect crucial infrastructure through strengthening the nodes which are chosen by the average degree strategy.

VI. CONCLUSION AND DISCUSSION

The study of the network disintegration is primarily to remove a group of nodes to achieve the maximum disintegration effect. Seeking an optimal solution among massive alternative strategies with the heterogeneous cost and cost constraint is a significant and challenging problem. Most related studies of network disintegration problem assumed that the disintegration cost is equal while neglecting a critical reality that the disintegration cost of each node (edge) might be heterogeneous. In this work, we concentrate on the network disintegration problem with heterogeneous disintegration cost, where the disintegration cost to eliminate each node might be non-identical.

We first put forward a disintegration cost model and an optimization model for disintegration strategy. Then, we analyze three typical strategies to explore their disintegration effects on synthetic networks and real-world networks. Numerical experiments indicate that the disintegration effect of hub strategy drops gradually when the disintegration cost changes from homogeneity to heterogeneity. For the situation of strong heterogeneity of disintegration cost, average degree strategy achieves the maximum disintegration effect gradually. Also, taking another perspective, average degree strategy may also enlighten efficient methods to protect crucial infrastructure through strengthening the nodes which are chosen by the average degree strategy. In the future study, we would try more suitable functions to define the disintegration effect first, such as weighted linear function and logarithmic function. Then, we would introduce more reasonable disintegration strategies to the problem of network disintegration, which deserves further investigation.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grant Nos. 71371185 and 71690233, and the Program for New Century Excellent Talents in University under Grant No. NCET-12-0141.

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